Fractal Models of Natural Phenomena

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Fractals
What is a Fractal?

- A complex object
- The complexity of which derives from *self-similarity*
- Or the repetition of form over a (finite) range of scales

"Bigger swirls have smaller swirls
that feed on their velocity,
and smaller swirls have smaller swirls
and so on, to viscosity"

Fractals
Fractal dimension

- Generalization of familiar integer-valued dimension
- Fractal dimension is real-valued, e.g., 3.3
- Large value after decimal point þ rough surface
- Small value after decimal point þ smoother surface
- Fractal dimension is not mathematically well-defined
- Can be used entirely subjectively

Fractals
Dilation symmetry

- The easiest way to think of fractals:
  - *Dilation symmetry*
  - Invariance under change of scale (zooming in and out)
- Symmetry may be *exact* or *statistical*
  - Exact self-similarity: Koch snowflake
  - Statistical self-similarity: terrains, clouds

Fractals
Deterministic vs. random fractals

- Deterministic fractals
  - Koch snowflake
  - Mandelbrot set
- Random fractals
  - Iterated function systems
  - L-systems
  - Fractional Brownian motion (fBm)

Fractals
Complexity in Nature

- Nature is complex
- Fractals capture some—but not all—of that complexity
- Examples of fractals:
  - Trees
  - Mountains
  - Turbulence: clouds, fire, smoke, astronomical jets
- Counterexample:
  - A battered old tennis shoe

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Fractional Brownian Motion (fBm)

What is it?

- Generalization of ***Brownian motion***:
  - Integral of progress on a random walk
- fBm is characterized by its ***power spectrum***
  - Brownian motion has $1/f^2$ power spectrum
  - fBm has $1/f^b$ power spectrum, 1.0 = b = 3.0
- Just think of b as controlling roughness of the terrain
- For math, see Voss & Saupe in “The Science of Fractal Images”

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Fractional Brownian Motion (fBm)

The key variables

- ***Basis function***:
  - The shape that is repeated over a range of scales
- ***Spectral exponent***:
  - Determines fractal dimension, or roughness of terrain
- ***Lacunarity***:
  - The gap between successive scales
- ***Octaves***:
  - The number of scales of self similarity

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Fractional Brownian Motion (fBm)

The basis function

- Should have range [-1.0 . . 1.0]
  - So that integral remains zero
  - Expected value remains zero
- Shape is very important
  - Shape clearly shows through in fractal sum
  - (At lacunarity of 2.0)
- Can be literally anything!
  - Sparse convolution (***wavelets***) gives maximum flexibility
  - But is very expensive
  - (See Peachey in “Textures and Modelling: A Procedural Approach”)

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Fractional Brownian Motion (fBm)

The basis function

- Sine wave in Fourier synthesis
  - Mathematically pure: each frequency is defined exactly
  - Sine is periodic, so all finite sums of it are also periodic
- Triangle wave in polygon subdivision
  - Piecewise linear interpolation
  - Creases and sharp peaks
- Perlin noise
  - Piecewise cubic interpolation
  - Nice, aperidoic sine-wave substitute
- Others
  - Voronoi (see Worley, SIGGRAPH 96)
  - See list of basis functions in MojoWorld in CAL

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Fractional Brownian Motion (fBm)

The basis function

- Batch algorithms
  - Fourier synthesis
  - Polygon subdivision
- Point-evaulated
  - Perlin noise, Voronoi, sparse convolution
  - These are the so-called “procedural” methods
- Infinite support
  - Sine waves
  - Procedural noises
- Finite support
  - Polygon subdivision
  - Wavelets

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Fractional Brownian Motion (fBm)

The spectral exponent

- Determines the fractal dimension
- Or the roughness of our terrain
- Can be used correctly or incorrectly
- But you get a fractal nonetheless
  - See the course notes for the math
  - And the literature for unexpected complications
- But don’t worry—use it qualitatively and ignore the math!

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Fractional Brownian Motion (fBm)

Lacunarity

- The gap between frequencies in spectral summation
- Virtually always set to 2.0 (hence “octaves”)
- May want to use 2.0 ± ~0.1
  - To avoid artifacts in lattice-based noises
  - As with value & gradient Perlin noises
- Using values << 2.0
  - Slower: takes more octaves to get fine details
  - Gains little, visually
- Using values >> 2.0
  - Faster: takes less octaves to get fine details
  - But discrete frequencies can show through

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Fractional Brownian Motion (fBm)

Octaves

- Number of octaves is number of scales of self similarity
- Octaves are only “octaves” when lacunarity is 2.0
- Octaves = detail
- Can be driven by Nyquist limit
  - To antialias by *clamping*
  - As in QAEB tracing (see “Textures and Modelling”)
  - Yields pixel-sized detail everywhere

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Fractal Terrain Models

Different kinds

- *Fourier*:
  - The most mathematically “pure”
  - Slow and periodic
- *Polygon subdivision*:
  - Easiest to implement, but sports the worst artifacts
  - Triangle, square, nested, unnested, semi-nested
  - (see Miller, SIGGRAPH 86, and “The Science of Fractal Images”)
- *Point-evaluated / procedural*:
  - Can be the slowest, depending on basis function
  - Most flexible and, generally, best-looking

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Fractal Terrain Models

Point-evaluated or procedural

- Perlin noise fBm
  - Generalization of Perlin’s “chaos” function
  - (See Perlin, *An Image Synthesizer,* SIGGRAPH 85)
- General procedural fBm
  - Same as above
  - But using Voronoi noise, sparse convolution, etc.
- Domain-distorted procedural fBm
  - Add a vector-valued function to the point, before evaluation
  - Shmushes the resulting fractal around

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Fractal Terrain Models

Point-evaluated or procedural

The basic algorithm:

1. Start with lowest frequency (largest scale of basis)
2. Double the frequency
3. Scale amplitude down, according to spectral exponent
4. Add in new, scaled frequency
5. Goto 2.

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Code for Procedural fBm

```
fBm( Vector point,
  NoiseFunction basis(),
  real exponent, real lacunarity,
  integer octaves )
{
  real value = 0.0, amplitude = 1.0;

  for ( i=0; i<octaves; i++ ) {
     value += basis(point) * amplitude;
     point *= lacunarity;
     amplitude *= exponent;
  }

  return value;
}
```

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Multifractals

Heterogeneous terrain models

- fBm is *stationary*: statistically homogeneous and isotropic
- Real terrain is far more complex
  - Mountains rise from plains
  - Peaks and valleys have different roughnesses, etc.
- We want to capture at least some of this
  - Devising heterogeneous fBm-based fractals
  - While preserving the elegance of fBm

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Multifractals

Three multifractal terrain models

- Stats-by-altitude
  - Conjecture: valleys are smoother than peaks
  - Model: multiply each octave (after first) by current "altitude"
- "Pure" multiplicative multifractal
  - Inner loop is multiplicative, rather than additive as in fBm
  - Problem: converges to zero or diverges to infinity
- Hybrid additive/multiplicative multifractal
  - Conjecture: valleys should be smoother at all altitudes
  - Model: multiply octave *i* by value of octave *i* -1; sum this

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Code for Stats-by-Altitude Multifractal

```
StatsByAlt( Vector point, NoiseFunction basis(),
        real exponent, real lacunarity,
         integer octaves )
{
    real value, amplitude = 1.0;

    if ( octaves )               // do first octave
          value = basis( point );

    for ( i=1; i<octaves; i++ ) {
          value += value * basis(point) * amplitude;
          point *= lacunarity;
          amplitude *= exponent;
    }

    return value;
}
```

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Code for Multiplicative Multifractal

```
Multifractal( Vector point,
          NoiseFunction basis(),
          real exponent, real lacunarity,
          integer octaves )
{
    real value = 1.0, amplitude = 1.0;

    for ( i=0; i<octaves; i++ ) {
          value *= basis(point) * amplitude;
          point *= lacunarity;
          amplitude *= exponent;
    }

    return value;
}
```

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Code for Hybrid Multifractal

```
HybridMF( Vector point, NoiseFunction basis(),
       real exponent, real lacunarity,
       integer octaves )
{
    real value, signal, weight, amplitude = 1.0;

    if ( octaves <= 0 ) return 0.0;

    weight = value = basis( point );          // first octave
    octaves -= 1.0;

    for ( i=1; i<octaves; i++ ) {
          signal = weight * basis(point) * amplitude;
          value += signal;
          weight = signal;
          point *= lacunarity;
          amplitude *= exponent;
    }

    return value;
}
```

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Erosion

- Erosion is what shapes terrains
  - Bedrock is fractal; erosion works on this fractal substrate
  - Creates *context-sensitive* fractals: river networks
- Diffusive erosion
  - Dry creep, rain splash, animal activity, etc.
  - Equivalent to low-pass filter—can operate very efficiently
- Fluvial erosion: running water
  - Rivers and glaciers are principal geomorphic agents
  - Very important—but too hard to implement and slow to run!
  - (see Musgrave et al, SIGGRAPH 89, and geology literature)

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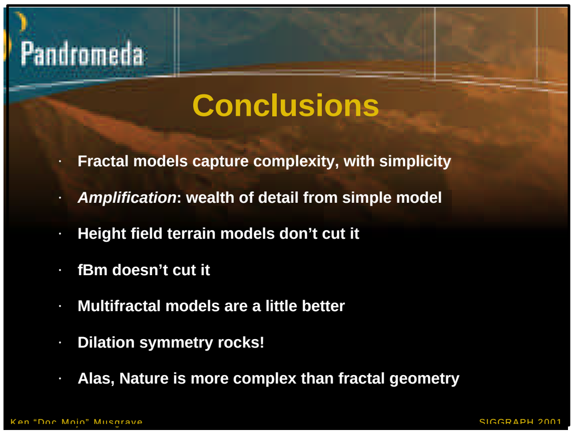
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Conclusions
Fractal models capture complexity, with simplicity
Amplification: wealth of detail from simple model
Height field terrain models don't cut it
fBm doesn't cut it
Multifractal models are a little better
Dilation symmetry rocks!
Alas, Nature is more complex than fractal geometry
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